Integer-Forcing: An Algebraic Approach to Interference Management

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Motivation
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Usual Assumptions:

- Each antenna carries an independent data stream $x_\ell \in \mathbb{C}^n$ of rate $R$ (e.g., V-BLAST setting, cellular uplink). $X = [x_1 \cdots x_M]^T$.
- Usual power constraint: $\|x_\ell\|^2 \leq \eta\text{SNR}$.
- Channel model: $Y = HX + Z$
- $Z$ is elementwise i.i.d. $\mathcal{CN}(0, 1)$.
- **CSIR**: Only the receiver knows channel realization $H \in \mathbb{C}^{M \times M}$. 
Throughout the talk, we will assume that $\mathbf{H}$ is elementwise i.i.d. Rayleigh, remains fixed throughout the block, and is only known at the receiver.

Say that we have a scheme that achieves rate $R_{\text{scheme}}(\mathbf{H})$ under channel realization $\mathbf{H}$. For a target rate $R$, the outage probability is

$$p_{\text{out}}(R) = \mathbb{P}(R_{\text{scheme}}(\mathbf{H}) < R)$$

and the outage rate is

$$R_{\text{out}}(\rho) = \sup \{ R : p_{\text{out}}(R) \leq \rho \}.$$
Joint Maximum Likelihood Decoding:

\[ R_{\text{joint}}(H) = \min_{S \subseteq \{1, \ldots, M\}} \frac{1}{|S|} \log \det (I + \text{SNR} \, H_S H_S^*) \]

- Corresponds to the (symmetric) outage capacity.
- Naive implementation has prohibitively high complexity.
- Of course, there are many clever ways to reduce the complexity!
Zero-Forcing and Linear MMSE Receivers:

- Project the received signal, $\tilde{Y} = BY$ to eliminate interference between data streams.
- After projection, single-user decoders attempt to recover the individual data streams.
- Optimal $B$ is the MMSE projection.
**MIMO Uplink Channel: Zero-Forcing and Linear MMSE**

**Zero-Forcing and Linear MMSE Receivers:**

- The $m^{th}$ SISO decoder tries to recover $x_m$ from $b_m^T Y$:

  $$\text{SINR}_{\text{LMMSE},m}(H) = \max_{b_m} \frac{\text{SNR} \|b_m^T h_m\|^2}{1 + \text{SNR} \sum_{\ell \neq m} \|b_m^T h_\ell\|^2}$$

- Rate per user:

  $$R_{\text{LMMSE}}(H) = \min_{m=1,\ldots,M} \log \left(1 + \text{SINR}_{\text{LMMSE},m}(H)\right)$$
Successive Interference Cancellation Receivers:

- Decode in order $\pi$. Cancel $x_{\pi(1)}, \ldots, x_{\pi(m-1)}$ from $\tilde{y}_m$:

$$\text{SINR}_{\text{SIC}, \pi(m)}(H) = \max_{b_m} \frac{\text{SNR} \|b_m^T h_{\pi(m)}\|^2}{1 + \text{SNR} \sum_{\ell=m+1}^M \|b_m^T h_{\pi(\ell)}\|^2}$$

- Rate per user:

$$R_{\text{V-BLAST II}}(H) = \max_{\pi} \min_{m=1,\ldots,M} \log \left(1 + \text{SINR}_{\text{SIC}, \pi(m)}(H)\right)$$
MIMO Uplink Channel: Integer-Forcing

What if we could decode something else?

- **Zero-Forcing / LMMSE:** First, eliminate interference. Then, decode individual data streams.
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First, decode
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- **Zero-Forcing / LMMSE**: First, eliminate interference. Then, decode individual data streams.

- **Integer-Forcing**: First, decode integer-linear combinations.
MIMO Uplink Channel: Integer-Forcing

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• **Integer-Forcing:** First, decode integer-linear combinations.
  Then, eliminate interference.
What if we could decode something else?

- **Zero-Forcing / LMMSE:** First, eliminate interference. Then, decode individual data streams.

- **Integer-Forcing:** First, decode integer-linear combinations. Then, eliminate interference.

- If the **integer matrix** $A$ is full rank, we can successfully recover the individual data streams.
Integer-Forcing Linear Receivers:

- The \( m^{th} \) effective channel after projection is

\[
b_m^T Y = b_m^T H X + b_m^T Z
\]
Integer-Forcing Linear Receivers:

- The $m^{th}$ effective channel after projection is

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$$= a_m^T X + (b_m^T H - a_m^T) X + b_m^T Z$$
**Integer-Forcing Linear Receivers:**

- The $m^{th}$ effective channel after projection is

$$b_m^T Y = b_m^T H X + b_m^T Z$$

$$= a_m^T X + (b_m^T H - a_m^T) X + b_m^T Z$$

$$= \sum_{\ell=1}^{M} a_m \ell x_{\ell}^T + (b_m^T H - a_m^T) X + b_m^T Z$$

- **Codeword**
- **Effective Noise**
Integer-Forcing Linear Receivers:

- The $m^{th}$ effective channel after projection is

\[
b_m^T Y = b_m^T HX + b_m^T Z = a_m^T X + (b_m^T H - a_m^T)X + b_m^T Z
\]

\[
= \sum_{\ell=1}^{M} a_{m\ell} x_\ell^T + \underbrace{(b_m^T H - a_m^T)X + b_m^T Z}_{\text{Effective Noise}}
\]

- The $a_{m\ell} \in \mathbb{Z}[j]$ are Gaussian integers and the codebook should be closed under integer-linear combinations.

- We are free to choose any full-rank integer-valued matrix $A$. 
**MIMO Uplink Channel: Integer-Forcing**

**Integer-Forcing Linear Receivers:** (Zhan-Nazer-Erez-Gastpar ’12)

- The \( m^{th} \) SISO decoder tries to recover \( \sum_\ell a_m \ell \mathbf{x}_\ell \) from \( \mathbf{b}_m^T \mathbf{Y} \):

\[
\text{SINR}_{IF,m}(\mathbf{H}, \mathbf{A}) = \max_{\mathbf{b}_m} \frac{\mathrm{SNR}}{\|\mathbf{b}_m\|^2 + \mathrm{SNR} \|\mathbf{b}_m^T \mathbf{H} - a_m^T\|^2}
\]

- Rate per user:

\[
R_{IF}(\mathbf{H}) = \max_{\mathbf{A}} \min_{m=1,\ldots,M} \log^+ \left( \text{SINR}_{IF,m}(\mathbf{H}, \mathbf{A}) \right)
\]

- Includes linear MMSE as a special case by setting \( \mathbf{A} = \mathbf{I} \).
Comparison: Outage Rates

2 users, 2 receive antennas, Rayleigh fading, 1% outage.
Comparison: Outage Rates

4 users symmetric rate case

- Capacity
- Integer-Forcing
- Linear MMSE

4 users, 4 receive antennas, Rayleigh fading, 1% outage.
Questions

• How can we efficiently select a good integer matrix $A$?

• How does the performance scale with the number of users?

• How sensitive is the performance to imperfect CSIR?

• What types of SISO encoders and decoders can we use?

• What about the downlink?

• Can we move beyond this idealized problem setting?
Finding a Good Integer Matrix

\[
\text{SINR}_{IF,m}(H, A) = \max_{b_m} \frac{\text{SNR}}{\|b_m\|^2 + \text{SNR} \|H^T b_m - a_m\|^2}
\]

- Optimal \(b_m\) is the MMSE projection.

- Plugging in and applying the Matrix Inversion Lemma, we get that

\[
\text{SINR}_{IF,m}(H, A) = \frac{1}{\| (I + \text{SNR} H^*H)^{-1/2} a_m \|^2}
\]

- Finding the optimal \(A\) corresponds to finding a good lattice basis.

- This is a hard problem in general but good approximation algorithms are known, such as the LLL algorithm.

- We are currently using a slight twist: We run LLL to get a lattice basis. Then, we turn to the dual lattice and run LLL again, initializing with the first basis.
Rayleigh fading, 1% outage.
How does the performance scale with the number of users?

20 dB symmetric rate case

- Capacity (Upper Bound)
- Integer-Forcing
- Linear MMSE

Rayleigh fading, 1% outage.
What is the impact of imperfect CSIR?

1. Receiver only sees $H + E$ where $E$ is elementwise i.i.d. $\mathcal{CN}(0, \sigma^2)$.
2. May result in selecting both a suboptimal integer matrix $A$ and a suboptimal projection matrix $B$. 
4 users, 20dB, Rayleigh fading, 1% outage.
What kinds of SISO coding schemes can be used?

- Underlying integer-forcing is the compute-and-forward framework, which is used as a black box to recover linear combinations of the messages over some finite field $\mathbb{F}_p$.

- Messages are vectors over a prime-sized finite field, $w_\ell \in \mathbb{F}_p^k$. 
• Architecture is completely digital after SISO decoders.
What kinds of SISO coding schemes can be used?

- **Nazer-Gastpar ’11**: Compute-and-forward achievability proofs via nested lattice codes.

- High-dimensional nested lattice codes lead to nice \( \log(\text{SINR}) \) expressions but have high implementation complexity.

- Remember, all we actually need is that the codebook is closed under integer-linear combinations.
What kinds of SISO coding schemes can be used?

- What about QAM combined with a binary linear code?

- **Issue:** Real addition does not map well to addition over \( \mathbb{F}_{2^M} \).

\[
[x_1 + x_2] \mod 2^M \neq x_1 \oplus x_2
\]
What kinds of SISO coding schemes can be used?

- What about $p$-ary QAM where $p$ is prime combined with a linear code over $\mathbb{F}_p$?

- Real addition maps well to addition over $\mathbb{F}_p$.

\[ [x_1 + x_2] \mod p = x_1 \oplus x_2 \]
Uncoded Integer-Forcing:

- Project by $b_m$, take $\mod p$, apply slicer.

- Correct if we recover $[a_{m_1}x_1 + a_{m_2}x_2 + \cdots + a_{m_M}x_M] \mod p$ for all $m$. 
Uncoded Integer-Forcing:

- Project by $b_m$, take $\mod p$, apply slicer.

- Correct if we recover $[a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mM}x_M] \mod p$ for all $m$.

- Is this lattice-aided reduction?
Uncoded Integer-Forcing:

- Project by $b_m$, take $\text{mod } p$, apply slicer.

- Correct if we recover $[a_{m_1}x_1 + a_{m_2}x_2 + \cdots + a_{m_M}x_M] \mod p$ for all $m$.

- Is this lattice-aided reduction? Nearly. We add the $\text{mod } p$. 
2 users, 2 receive antennas, Rayleigh fading

\( p = 13 \), fixed rate \( \frac{1}{2 \log(13)} \).
Coded Integer-Forcing:

- Project by $b_m$, take $\mod p$, apply LDPC decoding algorithm.

- Correct if we recover $[a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mM}x_M] \mod p$ for all $m$. 
2 users, 2 receive antennas, Rayleigh fading

\( p = 13 \), fixed rate \( \frac{1}{2} \log(13) \), regular \((3, 6)\) LDPC code.
• Lots of interesting questions on how to design low-complexity constellations and linear codes that work well for compute-and-forward.

• Several recent papers and...
Lots of interesting questions on how to design low-complexity constellations and linear codes that work well for compute-and-forward.

Several recent papers and...

Krishna’s talk coming up next!
• Capacity region is known. Requires dirty-paper coding. Caire-Shamai '03, Vishwanath-Jindal-Goldsmith '04, Viswanath-Tse '03, Yu-Cioffi '04, Weingarten-Steinberg-Shamai '06.
MIMO Downlink Channel: Zero-Forcing

Zero-Forcing Beamforming:

- Use beamforming matrix $\mathbf{B}$ to eliminate interference between data streams.
Integer-Forcing Beamforming: (Hong-Caire ’12,’13)

- Use beamforming matrix $\mathbf{B}$ to create an integer-valued effective channel $\mathbf{A}$.

- Decode linear combinations with $q_{m\ell} = [a_{m\ell}] \mod p$. 

$u_m = \bigoplus_{\ell} q_{m\ell} w_\ell$
**Integer-Forcing Beamforming:** (Hong-Caire ’12,’13)

- Use beamforming matrix $B$ to create an integer-valued effective channel $A$.

- Decode linear combinations with $q_{mℓ} = [a_{mℓ}] \mod p$.

Pre-invert $Q = [A] \mod p$ and decode messages.
**MIMO Downlink Channel: Integer-Forcing**

\[
\begin{bmatrix}
Q^{-1} \\
\vdots \\
Q^{-1}
\end{bmatrix}
\begin{bmatrix}
\tilde{w}_1 \\
\vdots \\
\tilde{w}_M
\end{bmatrix}
= \begin{bmatrix}
\tilde{x}_1 \\
\vdots \\
\tilde{x}_M
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_M
\end{bmatrix}
\begin{bmatrix}
y_1 \\
\vdots \\
y_M
\end{bmatrix}
= \begin{bmatrix}
\hat{w}_1 \\
\vdots \\
\hat{w}_M
\end{bmatrix}
\]

**Integer-Forcing Beamforming:** (Hong-Caire ’12,’13)

- Use beamforming matrix \( B \) to create an integer-valued effective channel \( A \).

- **Decode linear combinations** with \( q_{ml} = [a_{ml}] \mod p \).
  Pre-invert \( Q = [A] \mod p \) and decode messages.

- In very recent work, we have shown that uplink-downlink duality holds for integer-forcing. He-Nazer-Shamai ’14
Extensions

- What can we prove about the optimality of integer-forcing?
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• What about **space-time coding** at the transmitter?

• **Ordentlich-Erez ’13:** Linear dispersion codes + integer-forcing achieves the MIMO capacity universally to within a constant gap. Includes the optimal DMT as a special case.
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- What about successive cancellation for integer-forcing?

- Ordentlich-Erez-Nazer ’13: Framework for IF-SIC. Exact optimality if CSIT is available. Rate points tend to lie very close to the symmetric capacity.
Key Issues Going Forward

- Low-complexity constellations and codes.
- New algorithms for finding integer matrix $A$.
- Synchronization.
- What if the channel realization changes over the coding blocklength? (e.g., OFDM)
- How should we include rate adaptation?
- What does this mean for user selection?
- With Behnaam, Krishna, and students, we are working towards a WARP implementation.
- Any others?